

NON-TRIVIAL SINGULAR SPECTRAL SHIFT FUNCTIONS EXIST

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ABSTRACT. In this paper I prove existence of an irreducible pair of operators H_0 and $H_0 + V$, where H_0 is a self-adjoint operator and V is a self-adjoint trace-class operator, such that the singular spectral shift function $\xi_{H_0+V, H_0}^{(s)}$ of the pair is non-zero on the absolutely continuous spectrum of H_0 .

1. INTRODUCTION

Let H_0 be a self-adjoint operator and V be a trace-class self-adjoint operator. The Lifshits-Krein spectral shift function ξ is the unique L_1 -function on \mathbb{R} such that for any $f \in C^\infty(\mathbb{R})$ with compact support the trace formula

$$\mathrm{Tr}(f(H_0 + V) - f(H_0)) = \int f'(\lambda) \xi(\lambda) d\lambda$$

holds. Existence of such a function was proved in [Kr]. Further, in [BS], M. Sh. Birman and M. Z. Solomyak established the formula

$$\xi(\lambda) = \frac{d}{d\lambda} \int_0^1 \mathrm{Tr} (V E_{(-\infty, \lambda]}(H_r)) dr$$

for the spectral shift function; this formula is called spectral averaging formula.

In [BK], M. Sh. Birman and M. G. Krein proved that for a.e. $\lambda \in \mathbb{R}$ the formula

$$(1) \quad \det S(\lambda; H_0 + V, H_0) = e^{-2\pi i \xi(\lambda)}$$

holds, where $S(\lambda; H_0 + V, H_0)$ is the so-called scattering matrix of the pair $H_0 + V$ and H_0 (see e.g. [Y, Az₂]).

In [Az] I showed that for Shrödinger operators with sufficiently regular potentials the following variant of the Birman-Krein formula holds:

$$(2) \quad \det S(\lambda; H_0 + V, H_0) = e^{-2\pi i \xi^{(a)}(\lambda)}.$$

The function $\xi^{(a)}(\lambda)$, which in [Az] was called the absolutely continuous part of the spectral shift function, can be given by the formula

$$\xi^{(a)}(\lambda) := \frac{d}{d\lambda} \int_0^1 \mathrm{Tr} (V E_{(-\infty, \lambda]}(H_r^{(a)})) dr,$$

where $E_{(-\infty, \lambda]}(H)$ denotes the spectral projection of a self-adjoint operator H , $H_r := H_0 + rV$, $r \in \mathbb{R}$, and $H_r^{(a)}$ is the absolutely continuous part of H_r . The formula (2) compared with the Birman-Krein formula (1) obviously implies that the function

$$(3) \quad \xi^{(s)}(\lambda) := \frac{d}{d\lambda} \int_0^1 \text{Tr} (V E_{(-\infty, \lambda]}(H_r^{(s)})) dr,$$

called in [Az] the singular part of the spectral shift function (or just singular spectral shift function), is a.e. integer-valued. Here $H_r^{(s)}$ denotes the singular part of H_r . For Schrödinger operators from the class, considered in [Az], this is a trivial result: those operators do not have singular spectrum on the positive semi-axis, so that $\xi^{(s)} = 0$ on $[0, \infty)$, while on $(-\infty, 0]$ the function $\xi^{(s)}$ coincides with ξ , which is well-known to be integer-valued on $(-\infty, 0]$.

In [Az₂] the formula (2) was proved for arbitrary self-adjoint operators H_0 and arbitrary self-adjoint trace-class perturbations V . Combined with the Birman-Krein formula, this implies that for any such operators the singular spectral shift function is a.e. integer-valued. Looking at the definition (3) of the singular spectral shift function, this is quite an unexpected result (in my opinion). But still, there were no examples of non-trivial singular spectral shift functions.

By a non-trivial example of singular spectral shift function I mean an example of a self-adjoint operator H_0 and a self-adjoint trace-class operator V , such that H_0 and V do not have common non-trivial invariant subspace and such that the restriction of the singular spectral shift function for the pair $H_0 + V$ and H_0 to the absolutely continuous spectrum of H_0 is non-zero. Trivial examples are easily constructed: one can take the direct sum of any pair with a finite-dimensional pair, — if the support of the spectral shift function of the finite-dimensional pair overlaps the absolutely continuous spectrum of another pair, we are done.

In this paper I prove existence of a non-trivial singular spectral shift function. The proof clearly indicates that non-trivial singular spectral shift functions not only exist, — they are ubiquitous. At the same time, it is not easy to point out to a concrete pair H_0 and V with non-trivial singular spectral shift function, and there seems to be a reason for this.

2. RESULTS

2.1. Preliminaries. Let H be a self-adjoint operator on a (complex separable) Hilbert space \mathcal{H} and let $\{E_\lambda\}$ be its spectral decomposition. Recall that a vector $f \in \mathcal{H}$ is absolutely continuous (respectively, singular, pure point, singular continuous) with respect to H , if the measure with distribution function $\langle E_\lambda f, f \rangle$ is absolutely continuous (respectively, singular, pure point, singular continuous). The set of all absolutely continuous (respectively, singular, pure point, singular continuous) vectors form a (closed) subspace of \mathcal{H} , which we denote by $\mathcal{H}^{(a)}$ (respectively, $\mathcal{H}^{(s)}$, $\mathcal{H}^{(pp)}$, $\mathcal{H}^{(sc)}$). The subspaces $\mathcal{H}^{(a)}$, $\mathcal{H}^{(s)}$, $\mathcal{H}^{(pp)}$, $\mathcal{H}^{(sc)}$ are mutually orthogonal and invariant with respect to H ; also, the

following decompositions hold: $\mathcal{H} = \mathcal{H}^{(a)} \oplus \mathcal{H}^{(s)}$ and $\mathcal{H}^{(s)} = \mathcal{H}^{(pp)} \oplus \mathcal{H}^{(sc)}$. Restriction of H to $\mathcal{H}^{(a)}$ (respectively, to $\mathcal{H}^{(s)}$, $\mathcal{H}^{(pp)}$, $\mathcal{H}^{(sc)}$) will be denoted by $H^{(a)}$ (respectively, by $H^{(s)}$, $H^{(pp)}$, $H^{(sc)}$). The equalities

$$H = H^{(a)} \oplus H^{(s)} \quad \text{and} \quad H^{(s)} = H^{(pp)} \oplus H^{(sc)}$$

hold. See e.g. [Y] for details.

Let H_0 be a self-adjoint operator and let V be a trace-class self-adjoint operator. We introduce the Lifshits-Krein spectral shift function ξ_{H_0+V, H_0} of the pair $H_0, H_0 + V$ by the Birman-Solomyak spectral averaging formula, as a distribution,

$$\xi(\varphi) = \int_0^1 \text{Tr}(V\varphi(H_r)) dr, \quad \varphi \in C_c^\infty,$$

where $H_r := H_0 + rV$, $r \in \mathbb{R}$. The distribution ξ is an absolutely continuous finite measure (see [Kr]). Similarly, we introduce the absolutely continuous $\xi_{H_0+V, H_0}^{(a)}$ and singular $\xi_{H_0+V, H_0}^{(s)}$ parts of the spectral shift function by formulas

$$\xi^{(a)}(\varphi) = \int_0^1 \text{Tr}(V\varphi(H_r^{(a)})) dr, \quad \xi^{(s)}(\varphi) = \int_0^1 \text{Tr}(V\varphi(H_r^{(s)})) dr,$$

where $H^{(a)}$ and $H^{(s)}$ denote the absolutely continuous and singular parts of a self-adjoint operator H respectively. It is shown in [Az₂] that $\xi^{(a)}$ and $\xi^{(s)}$ are also absolutely continuous finite measures (see [Az₂, Lemma 9.7]). Plainly,

$$\xi_{H_0+V, H_0} = \xi_{H_0+V, H_0}^{(a)} + \xi_{H_0+V, H_0}^{(s)}.$$

One can also introduce the pure point and singular continuous parts of the spectral shift function, by similar formulas:

$$\xi^{(pp)}(\varphi) = \int_0^1 \text{Tr}(V\varphi(H_r^{(pp)})) dr, \quad \xi^{(sc)}(\varphi) = \int_0^1 \text{Tr}(V\varphi(H_r^{(sc)})) dr.$$

Plainly,

$$\xi_{H_0+V, H_0}^{(s)} = \xi_{H_0+V, H_0}^{(pp)} + \xi_{H_0+V, H_0}^{(sc)}.$$

If $V \geq 0$, then it is easy to see that $\xi^{(pp)}$ and $\xi^{(sc)}$ are also absolutely continuous. Whether they are absolutely continuous for arbitrary trace-class V , I don't know.

In [Kr] it was proved that ξ is additive; that is, for any three self-adjoint operators H_0, H_1, H_2 with common domain and trace-class differences the equality

$$\xi_{H_2, H_0} = \xi_{H_2, H_1} + \xi_{H_1, H_0}$$

holds. In [Az₃] it was proved that $\xi^{(a)}$ and $\xi^{(s)}$ are also additive (see [Az₃, Theorem 2.2 and Corollary 2.3]):

$$\xi_{H_2, H_0}^{(a)} = \xi_{H_2, H_1}^{(a)} + \xi_{H_1, H_0}^{(a)}, \quad \xi_{H_2, H_0}^{(s)} = \xi_{H_2, H_1}^{(s)} + \xi_{H_1, H_0}^{(s)}.$$

2.2. Idea of the proof. In case of operators H_0 and $H_0 + V$ on a finite dimensional Hilbert space, the spectral shift function $\xi(\lambda)$ is equal to the total number of eigenvalues of the family of operators $H_r = H_0 + rV$, which cross λ in the right direction minus the total number of eigenvalues which cross λ in the left direction. In finite-dimensional situation there is only pure point spectrum, so obviously $\xi(\lambda) = \xi^{(s)}(\lambda)$. Now, we take any other infinite-dimensional pair of operators H'_0 and $H'_0 + V'$ and take the direct sum of those two pairs: $H_0 \oplus H'_0$ and $(H_0 + V) \oplus (H'_0 + V')$. Obviously,

$$\xi_{(H_0+V) \oplus (H'_0+V'), H_0 \oplus H'_0}^{(s)} = \xi_{H_0+V, H_0}^{(s)} + \xi_{H'_0+V', H'_0}^{(s)},$$

with analogous equalities for all other parts of the spectral shift function. In this way we get a pair of infinite-dimensional operators H_0 and H_1 with non-zero singular spectral shift function, but the problem is that this pair is not irreducible. By the additivity property of the singular spectral shift function, for any other self-adjoint H_2 such that $H_2 - H_0$ is trace-class, we have the equality

$$\xi_{H_1, H_0}^{(s)} = \xi_{H_1, H_2}^{(s)} + \xi_{H_2, H_0}^{(s)}.$$

It follows that if $\xi_{H_1, H_0}^{(s)}$ does not vanish on the absolutely continuous spectrum $\sigma^{(a)}$ of H_0 , then at least one of the functions $\xi_{H_1, H_2}^{(s)}$ and $\xi_{H_2, H_0}^{(s)}$ also does not vanish on $\sigma^{(a)}$. (Note that since differences $H_i - H_j$ are trace-class, their absolutely continuous spectra coincide). Now, one has to ensure that both pairs (H_0, H_2) and (H_1, H_2) are either irreducible or that $\xi^{(s)}$ of an irreducible part of a pair does not vanish on the absolutely continuous spectrum of the reduced operators.

In accomplishing of this plan we have a big freedom of choice of operators due to the additivity property of the singular spectral shift function.

2.3. Results. Let $\tilde{\mathcal{H}} = L_2(\mathbb{R})$ and let $\mathcal{H} = \tilde{\mathcal{H}} \oplus \mathbb{C}$.

An operator A on \mathcal{H} can be represented in the form of a matrix

$$A = \begin{pmatrix} \tilde{A} & f_1 \\ \langle f_2, \cdot \rangle & z_0 \end{pmatrix},$$

where \tilde{A} is an operator on $\tilde{\mathcal{H}}$, $f_1, f_2 \in \tilde{\mathcal{H}}$ and $z_0 \in \mathbb{C}$. It is not difficult to see that A is self-adjoint if and only if \tilde{A} is self-adjoint, $f_1 = f_2$ and z_0 is real. So, a self-adjoint operator on \mathcal{H} has the form

$$H = \begin{pmatrix} D & f \\ \langle f, \cdot \rangle & a_0 \end{pmatrix},$$

where $D = D^*$ and $a_0 \in \mathbb{R}$.

Let $a \in \mathbb{R}$ and $v \in \tilde{\mathcal{H}}$. The self-adjoint operator

$$(4) \quad V_a = \begin{pmatrix} v \langle v, \cdot \rangle & v \\ \langle v, \cdot \rangle & a \end{pmatrix}$$

has rank less or equal to 2. Further, let

$$(5) \quad D = \frac{1}{i} \frac{d}{dx};$$

the operator D is a self-adjoint operator on $\tilde{\mathcal{H}}$, its spectrum is absolutely continuous and is equal to \mathbb{R} . Let also

$$(6) \quad H_0 = \begin{pmatrix} D & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

In this case $H_r := H_0 + rV$ is equal to

$$(7) \quad H_r = \begin{pmatrix} D & 0 \\ 0 & -1 + 2r \end{pmatrix}.$$

It is not difficult to see that

$$\xi_{H_1, H_0} = \xi_{H_1, H_0}^{(s)} = \chi_{[-1, 1]}.$$

Let $\tilde{V} = v \langle v, \cdot \rangle$, where from now on the vector $v \in \tilde{\mathcal{H}} = L_2(\mathbb{R})$ is given by

$$(8) \quad v(x) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{x^2}{2}};$$

the coefficient is chosen so that $\|v\| = 1$. It is well known that the Fourier transform of v is equal to v . The operator \tilde{V} is one-dimensional and so it is trace-class.

We say that a (closed) subspace \mathcal{K} of \mathcal{H} is invariant for an operator H on a Hilbert space \mathcal{H} , if $\mathcal{K} \cap \mathcal{D}(H)$ is dense in \mathcal{K} and if $Hf \in \mathcal{K}$ for all $f \in \mathcal{K} \cap \mathcal{D}(H)$. We say that a pair of operators H_1 and H_2 is irreducible, if the only subspaces of \mathcal{H} which are invariant for both H_1 and H_2 are $\{0\}$ and \mathcal{H} .

Lemma 2.1. *The pair (D, \tilde{V}) is irreducible.*

Proof. Let \mathcal{K} be a subspace of $\tilde{\mathcal{H}}$ which is invariant with respect to both D and \tilde{V} and let $\mathcal{K} \neq \mathcal{H}$.

(A) Claim: $v \notin \mathcal{K}$. It is known that the vector v is cyclic for D — Hermite polynomials which form a basis of $\tilde{\mathcal{H}}$ are linear combinations of $D^k v$. It follows that $v \notin \mathcal{K}$.

(B) Claim: $v \perp \mathcal{K}$. Indeed, if $f \in \mathcal{K}$ then $Vf = \langle v, f \rangle v \in \mathcal{K}$. If additionally $f \neq 0$ and if v is not orthogonal to f , then this implies that $v \in \mathcal{K}$. This contradicts (A).

(C) Claim: for any $n = 0, 1, 2, \dots$ $D^n v \perp \mathcal{K}$.

Proof. By (B), for $n = 0$ this is true. Assume that $D^n v \perp \mathcal{K}$ for $n = k$. Let $f \in \mathcal{K} \cap \mathcal{D}(D)$. Then $Df \in \mathcal{K}$, so by the assumption $\langle Df, D^k v \rangle = 0$. It follows that $\langle f, D^{k+1} v \rangle = 0$. Since $\mathcal{K} \cap \mathcal{D}(D)$ is dense in \mathcal{K} , this implies that $D^{k+1} v \perp \mathcal{K}$. The proof of (C) is complete.

(D) Since the vector v is cyclic for D , it follows from (C) that $\mathcal{K} = \{0\}$.

The proof is complete. \square

We consider operators V_1 and V_{-1} , given by (4).

It is not difficult to check that V_1 has rank one with eigenvector $\begin{pmatrix} v \\ 1 \end{pmatrix}$ and eigenvalue 2, and that V_{-1} has rank two with eigenvectors

$$\begin{pmatrix} v \\ -1 \pm \sqrt{2} \end{pmatrix}$$

and eigenvalues $\pm\sqrt{2}$. Note also, that

$$H_0 + V_1 = H_1 + V_{-1}.$$

Lemma 2.2. *Let H_0 be the operator (6), where D is given by (5), and let V_1 be the operator given by (4), where v is given by (8). Let $\mathbf{v} = \begin{pmatrix} v \\ 1 \end{pmatrix}$. Let \mathcal{K} be the subspace of $\tilde{\mathcal{H}}$ generated by vectors $H_0^k \mathbf{v}$, $k = 0, 1, 2, \dots$ and let $H' := H_0|_{\mathcal{K}}$ and $V' := V_1|_{\mathcal{K}}$ be restrictions of operators H_0 and V_1 to \mathcal{K} (it is easily seen that \mathcal{K} is invariant with respect to both H_0 and V_1). The pair (H', V') is irreducible.*

Proof. Assume the contrary: let \mathcal{L} be a non-trivial subspace of \mathcal{K} which is invariant with respect to both H_0 and V_1 . So, let \mathbf{f} be a non-zero vector from \mathcal{L} .

If $V_1 \mathbf{f}$ is non-zero, then the vector \mathbf{v} belongs to \mathcal{L} , since $V_1 \mathbf{f} = \alpha \mathbf{v}$ for some $\alpha \in \mathbb{C}$. Since \mathbf{v} is cyclic for H' (by definition), it follows that $\mathcal{L} = \mathcal{K}$. This shows that $V_1 \mathbf{f} = 0$ for any $\mathbf{f} \in \mathcal{L}$.

Since \mathcal{L} is invariant with respect to H' , the subspace $\mathcal{L} \cap \mathcal{D}(H')$ is dense in \mathcal{L} . Let $\tilde{\mathcal{L}}$ be the maximal domain of the restriction of H' to \mathcal{L} .

Now, let $\mathbf{f} \in \bigcap_{k=1}^{\infty} \mathcal{D}(H'|_{\tilde{\mathcal{L}}})^k$ be a non-zero vector (such a vector exists). The vector \mathbf{f} can be assumed to be either in the form $\begin{pmatrix} f \\ 0 \end{pmatrix}$ or $\begin{pmatrix} f \\ 1 \end{pmatrix}$.

If $\mathbf{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$, then $H_0^k \mathbf{f} = \begin{pmatrix} D^k f \\ 0 \end{pmatrix} \in \mathcal{L}$ for all $k = 0, 1, 2, \dots$; since V_1 vanishes on \mathcal{L} , it follows that $V_1(D^k f, 0) = 0$ for all k ; this implies that $\langle v, D^k f \rangle = 0$, which yields $\langle D^k v, f \rangle = 0$; since v is cyclic, it follows that $f = 0$. This is a contradiction.

So, let $\mathbf{f} = \begin{pmatrix} f \\ 1 \end{pmatrix}$. Since all vectors $H'^k \mathbf{f}$ belong to \mathcal{L} , it follows that all vectors

$$\begin{pmatrix} D^k f \\ (-1)^k \end{pmatrix}$$

belong to the kernel of V_1 . It follows from $V_1 \begin{pmatrix} D^k f \\ (-1)^k \end{pmatrix}$ that

$$\langle D^k v, f \rangle = \langle v, D^k f \rangle = (-1)^{k+1}$$

for all $k = 0, 1, 2, \dots$. It follows that

$$\langle D^k v, (1 + D)f \rangle = 0$$

for all $k = 0, 1, 2, \dots$. Since v is cyclic for D , it follows that $(1 - D)f = 0$. This implies that $f = 0$. This implies that $(0, 1) \in \tilde{\mathcal{L}}$, which in its turn implies that $V_1(0, 1) = 0$, so that $v = 0$. This is a contradiction.

The proof is complete. \square

Lemma 2.3. *Let H_1 be the operator (7) (with $r = 1$), where D is given by (5); let V_{-1} be the operator given by (4) with $a = -1$, where v is given by (8). The pair (H_1, V_{-1}) is irreducible.*

Proof. (A) Assume the contrary: let \mathcal{K} be a non-trivial subspace of \mathcal{H} , which is invariant with respect to both H_1 and V_{-1} .

Note that the proof will be complete, if we show that one of the vectors $\begin{pmatrix} v \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ belongs to \mathcal{K} . Indeed, in this case the invariance of \mathcal{K} with respect to V_{-1} would imply that the other vector also belongs to \mathcal{K} , and cyclicity of v with respect to D would complete the proof.

(B) Let $\mathbf{f} = \begin{pmatrix} f \\ f_0 \end{pmatrix} \in \mathcal{K}$ be a non-zero vector. Claim: we can assume that $f_0 = 1$. Indeed, assume the contrary: $\mathbf{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$. Since \mathcal{K} is invariant with respect to H_1 , it follows that $\begin{pmatrix} D^k f \\ 0 \end{pmatrix} \in \mathcal{K}$ for all $k = 0, 1, \dots$. Since also $V_{-1}\mathcal{K} \subset \mathcal{K}$, this implies that $\langle v, D^k f \rangle = 0$. It follows that $\langle D^k v, f \rangle = 0$ for all $k = 0, 1, \dots$. Since v is cyclic for D , it follows that $f = 0$. The claim is proved.

(C) So, let $\mathbf{f} = \begin{pmatrix} f \\ 1 \end{pmatrix} \in \mathcal{K}$. It follows that

$$V_{-1}\mathbf{f} = \begin{pmatrix} (\langle v, f \rangle + 1)v \\ \langle v, f \rangle - 1 \end{pmatrix} \in \mathcal{K}.$$

If one of the numbers $\langle v, f \rangle \pm 1$ is zero, then the proof is complete by (A).

So, we can assume that $\langle v, f \rangle \pm 1 \neq 0$.

It follows that $\begin{pmatrix} v \\ \alpha \end{pmatrix} \in \mathcal{K}$, where $\alpha = \frac{\langle v, f \rangle - 1}{\langle v, f \rangle + 1}$. Since $V_{-1}\begin{pmatrix} v \\ \alpha \end{pmatrix} = (1 + \alpha)^{-1} \begin{pmatrix} v \\ \frac{1 - \alpha}{1 + \alpha} \end{pmatrix} \in \mathcal{K}$ too, if $\alpha \neq -1 \pm \sqrt{2}$ (that is, if $\begin{pmatrix} v \\ \alpha \end{pmatrix}$ is not an eigenvector of V_{-1}), it follows that $\begin{pmatrix} v \\ 0 \end{pmatrix} \in \mathcal{K}$, and (A) completes the proof.

So, we can assume that $\begin{pmatrix} v \\ \alpha \end{pmatrix} \in \mathcal{K}$ is an eigenvector of V_{-1} . Since $H_1\begin{pmatrix} v \\ \alpha \end{pmatrix} \in \mathcal{K}$, it follows that $\begin{pmatrix} Dv \\ \alpha \end{pmatrix} \in \mathcal{K}$. Further,

$$V_{-1}\begin{pmatrix} Dv \\ \alpha \end{pmatrix} = \begin{pmatrix} (\langle v, Dv \rangle + \alpha)v \\ \langle v, Dv \rangle - \alpha \end{pmatrix} \in \mathcal{K}.$$

It is easy to check that this vector and $\begin{pmatrix} v \\ \alpha \end{pmatrix} \in \mathcal{K}$ are linearly independent; hence, the proof is complete by (A). \square

Theorem 2.4. *There exists an irreducible pair consisting of a self-adjoint operator H_0 with non-empty absolutely continuous spectrum and a trace-class self-adjoint operator*

V , such that the restriction of the singular spectral shift function $\xi_{H_0+V, H_0}^{(s)}$ of the pair $(H_0, H_0 + V)$ to the absolutely continuous spectrum of H_0 is non-zero.

Proof. Let H_0 , V_1 and V_{-1} be as in Lemmas 2.2 and 2.3. Additivity of the singular spectral shift [Az₃, Corollary 2.3] implies that

$$(9) \quad \xi_{H_0+V_1, H_0}^{(s)} + \xi_{H_1, H_1+V_{-1}}^{(s)} = \xi_{H_1, H_0}^{(s)} = \chi_{[-1, 1]}.$$

So, at least one of the functions $\xi_{H_0+V_1, H_0}^{(s)}$ or $\xi_{H_1, H_1+V_{-1}}^{(s)}$ is non-zero on $[-1, 1]$, which is a subset of the absolutely continuous spectrum of H_0 (and of H_1).

If $\xi_{H_1, H_1+V_{-1}}^{(s)} \neq 0$ on $[-1, 1]$, then we are done by Lemma 2.3.

If $\xi_{H_1, H_1+V_{-1}}^{(s)} = 0$ on $[-1, 1]$, then we note that the complement of the subspace \mathcal{K} generated by vectors $H_0^k \mathbf{v}$, $k = 0, 1, 2, \dots$, has dimension ≤ 1 ; the subspace \mathcal{K} is invariant with respect to both H_0 and V_1 , and by Lemma 2.3 the restrictions H'_0 and V'_1 of operators H_0 and V_1 to the subspace \mathcal{K} give an irreducible pair. Now note that the restriction of V_1 to \mathcal{K}^\perp is zero, so that

$$\xi_{H'_0+V'_1, H'_0}^{(s)} = \xi_{H_0+V_1, H_0}^{(s)} = \chi_{[-1, 1]}.$$

Further, since \mathcal{K}^\perp is at most one-dimensional, the difference $H_0 - H'_0$ is a finite-rank operator. This implies that the absolutely continuous spectrum of H'_0 coincides with that of H_0 , which is \mathbb{R} .

The proof is complete. \square

Lemma 2.5. *If $r > 0$, then for any $\alpha \in \mathbb{R}$ the pure point spectrum of the operator*

$$H := \begin{pmatrix} D + r \langle v, \cdot \rangle v & rv \\ r \langle v, \cdot \rangle & \alpha \end{pmatrix},$$

where D is given by (5) and v is given by (8), is empty.

Proof. Assume that there is a non-zero vector $\mathbf{f} = \begin{pmatrix} f \\ f_0 \end{pmatrix} \in \mathcal{H}$ such that $H\mathbf{f} = \lambda\mathbf{f}$ for some $\lambda \in \mathbb{R}$. This implies that f belongs to the domain of D . Further, we have

$$H\mathbf{f} = \begin{pmatrix} Df + r \langle v, f \rangle v + rf_0v \\ r \langle v, f \rangle + \alpha f_0 \end{pmatrix} = \begin{pmatrix} \lambda f \\ \lambda f_0 \end{pmatrix}.$$

This implies that

$$Df = \lambda f - r \langle v, f \rangle v - rf_0v,$$

so that $f' \in L_2(\mathbb{R})$. Taking the Fourier transform of the last equality gives

$$\xi \hat{f}(\xi) = \lambda \hat{f}(\xi) - r \langle v, f \rangle \hat{v}(\xi) - rf_0 \hat{v}(\xi).$$

Since $\hat{v} = v$, it follows that

$$\hat{f}(\xi) = -r (\langle v, f \rangle + f_0) \cdot \frac{v(\xi)}{\xi - \lambda}.$$

Since $\frac{v(\xi)}{\xi - \lambda}$ is not L_2 , it follows that $f = 0$ and $f_0 = 0$; that is, $\mathbf{f} = 0$.

This contradiction completes the proof. \square

By Lemma 2.5, the pure point spectrum of operators H_0+rV_1 and H_1+rV_{-1} , $r \in (0, 1]$, is empty. It follows that for both pairs of operators

$$(10) \quad \xi_{H_0+rV_1, H_0}^{(pp)} = 0 \quad \text{and} \quad \xi_{H_1, H_1+rV_{-1}}^{(pp)} = 0.$$

Consequently, by (9),

$$\xi_{H_0+V_1, H_0}^{(sc)} + \xi_{H_1, H_1+V_{-1}}^{(sc)} = \chi_{[-1, 1]},$$

so that

$$\xi_{H_0+V_1, H_0}^{(sc)} \neq 0 \quad \text{or} \quad \xi_{H_1, H_1+V_{-1}}^{(sc)} \neq 0.$$

Corollary 2.6. *The pure point and the singular continuous spectral shift distributions are not additive.*

Proof. It follows from (10) and (9) that pure point spectral shift function is not additive. This fact, combined with additivity of the singular spectral shift function, implies non-additivity of the singular continuous spectral shift function. \square

This corollary shows a significant difference between absolutely continuous and singular spectral shift functions, on one hand, and pure point and singular continuous spectral shift functions on the other hand.

Note that the results of this paper are in full accordance with the generic property of the singular continuous spectrum which was observed in [RJMS, RMS] (see also [S] and [SW]).

The example of a non-trivial singular spectral shift function presented here is a clear indication of the fact that non-trivial singular spectral shift functions are ubiquitous, even though it is not easy to present them explicitly; they are a bit like transcendental numbers.

Note that eigenvalues of H_r in the absolutely continuous spectrum can pop up suddenly and disappear in the same way as r changes (see some examples in [S]), but it seems that they do not contribute to the singular part of the spectral shift function. In [Az₂] I made a conjecture that in the case of trace-class perturbations V the pure point part of the spectral shift function must be zero on the absolutely continuous spectrum (which is the same for all operators H_r in the path).

REFERENCES

- [Az] N. A. Azamov, *Infinitesimal spectral flow and scattering matrix*, preprint, arXiv:0705.3282v4.
- [Az₂] N. A. Azamov, *Absolutely continuous and singular spectral shift functions*, preprint, arXiv:submit/0092981v4.
- [Az₃] N. A. Azamov, *Singular spectral shift is additive*, arXiv:1008.2847v1.
- [BK] M. Sh. Birman, M. G. Kreĭn, *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR **144** (1962), 475–478.
- [BS] M. Sh. Birman, M. Z. Solomyak, *Remarks on the spectral shift function*, J. Soviet math. **3** (1975), 408–419.

- [Kr] M. G. Kreĭn, *On the trace formula in perturbation theory*, Mat. Sb., **33** 75 (1953), 597–626.
- [RJMS] R. del Rio, S. Jitomirskaya, N. Makarov, B. Simon, *Singular continuous spectrum is generic*, Bull. Amer. Math. Soc. **31** (1994), 208–212.
- [RMS] R. del Rio, N. Makarov, B. Simon, *Operators with singular continuous spectrum*, Comm. Math. Phys. **165** (1994), 59–67.
- [S] B. Simon, *Trace ideals and their applications: Second Edition*, Providence, AMS, 2005, Mathematical Surveys and Monographs, **120**.
- [SW] B. Simon, T. Wolff, *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*, Comm. Pure Appl. Math. **39** (1986), 75–90.
- [Y] D. R. Yafaev, *Mathematical scattering theory: general theory*, Providence, R. I., AMS, 1992.

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